In the second form of (15), the effect of depletion is accounted for explicitly in the leading term by an appropriate change of the expansion parameter. Numerical solutions of (7) for cesium and sodium ionization,[5] which were based on five ionized trace species (NO^+, Cs^+, or Na^+, O_2^-, O^-, e^-), five neutral air species (O_2, N_2, O, N, NO), and a ten-reaction system for the ionization process, were compared with (15). The comparison showed that the leading term of the second form of (15) agreed within 5% with all the results of the more elaborate calculations over a range of 0 to 99.5% ionization. Explicitly

$$\frac{C_m{}^+{}_{\max}}{C^m(\eta_{\max})} = \left\{1.0 + \frac{3lKu_e}{Sc\rho x(k/M)_{\text{eff}}}\right\}^{-1.0} \qquad (16)$$

for both cesium and sodium. The effective forward ionization rate in (16) combines three processes: a general neutral particle, O_2, and O.

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An Extension of Hetényi's Method through the use of Macaulay Brackets

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THERE seems to be a continuing interest in singularity functions as applied to beam and plate bending. In this journal, a recent article by Urry[1] extends the theory to beam columns, and a more comprehensive survey of the field by Weissenburger[2] traces the historical development of the subject. It should be noted in passing, that the work of Csonka[3, 4] treats the very general problem of an ordinary differential equation with constant coefficients whose right-hand side is composed of linear combinations of singularity functions of the Macaulay type. The example given by Urry can be handled as a special case by the method developed by Csonka, who solves this particular problem as an example. More recently Pilkey[5] has given an excellent account of the engineering significance of the method in an article that contains an exhaustive bibliography on the subject. Pilkey[6] has also introduced the theory of distribution of Laurent Schwartz to set the concept of singularity functions as used in bending theory on a sound mathematical basis. The purpose of this note is to extend the work of this writer[7–9] to the method of Hetényi.[10]

An example will serve to illustrate the application of the method. Consider a beam cantilevered at A, of total length $2a$, such that $\overline{AB} = \overline{BC} = a$. Let the loading on the beam be given as

$$0 < x < a \qquad w(x) = 0$$

$$a < x < 2a \qquad w(x) = (wo/a)(x - a)$$

where the origin is at A, and the end C is free.

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Following Hetényi, the deflection curve of part AB of the beam can be expanded into the following Maclaurin series:

$$y_{AB}(x) = y(0) + y'(0)x + y''(0)x^2/2! + \ldots + y^{(n)}(0)x^n/n!$$

The expansion being valid in the range $0 < x < a$.

Similarly, a Taylor expansion gives the deflection curve of part BC as

$$y_{BC}(x) = y(a) + y'(a)(x - a) + \ldots + y^{(n)}(a)(x - a)^n/n!$$

which is valid in $a < x < 2a$.

The Bernoulli-Euler equation gives

$$y''(x) = [M(x)/EI]$$

from which it follows that

$$y'''(x) = [V(x)/EI] \qquad y^{\text{IV}}(x) = [w(x)/EI] \qquad \text{etc.}$$

recalling the relations

$$dM/dx = V \qquad dV/dx = w$$

The Maclaurin expansion now reduces to

$$y_{AB}(x) = (M_A x^2/2EI) + (V_A x^3/6EI)$$

after the initial conditions $y(0) = y'(0) = 0$ have been inserted.

The Taylor expansion now becomes

$$y_{BC}(x) = y(a) + y'(a)(x - a) + \frac{M_B}{2EI}(x - a)^2 + \frac{V_B}{6EI}(x - a)^3 + \frac{w_0}{120\,EIa}(x - a)^5$$

where V_A, M_A, V_B, M_B, and w are assumed to act in the positive direction.

Equilibrium of the section AB of the beam requires that

$$M_A + V_A\,a = M_B \text{ and } V_A = V_B$$

The continuity of the slopes and deflections at B requires that

$$y'_{AB}(a) = y'_{BC}(a) \text{ and } y_{AB}(a) = y_{BC}(a)$$

This yields

$$y_{AB}(a) = \frac{M_A a^2}{2EI} + \frac{V_A a^3}{6EI} = y_{BC}(a)$$

$$y'_{AB}(a) = \frac{M_A a}{EI} + \frac{V_A a^2}{2EI} = y'_{BC}(a)$$

Rewriting the Taylor series after substituting the equilibrium and compatibility conditions leads to

$$y_{BC}(x) = \frac{M_A a^2}{2EI} + \frac{V_A a^3}{6EI} + \left(\frac{M_A a}{EI} + \frac{V_A a^2}{2EI}\right)(x - a) + \frac{(M_A + V_A a)}{2EI}(x - a)^2 + \frac{V_A}{6EI}(x - a)^3 + \frac{w_0}{120EIa}(x - a)^5$$

which, after some algebra, reduces to

$$y_{BC}(x) = \frac{M_A x^2}{2EI} + \frac{V_A x^3}{6EI} + \frac{w_0}{120\,EIa}(x - a)^5$$

This in turn can be written as

$$y_{BC}(x) = y_{AB}(x) + (w_0/120\,EIa)(x - a)^5$$

It is now clear that the use of Macaulay brackets can now be introduced to write

$$y(x) = \frac{M_A x^2}{2EI} + \frac{V_A x^3}{6EI} + \frac{w_0}{120\,EIa}\langle x - a\rangle^5$$

which is an expression now valid for the entire beam. This expression could have been written by inspection, starting with the Taylor expansion where the Maclaurin series were cut off, thus bypassing the equilibrium and compatibility conditions at B. It is obvious that M_A and V_A can be obtained by equilibrium considerations.

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Nonequilibrium Electrical Conductivity Measurements in Argon and Helium Seeded Plasmas

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In a previous paper,[1] the authors presented experimental values of electrical conductivity measured in a plasma composed of argon gas seeded with potassium vapor. The measurements were made at atmospheric pressure with a neutral gas temperature of 2000° ± 100°K and with a number of values of seed concentration in the range 0.2 to 0.8 mole %. The effect of nonequilibrium heating of the electron gas-excited potassium system was investigated for a range of current densities between 0.8 and 80 amp/cm². These data were in good agreement with values of the conductivity calculated by a scheme, outlined in Ref. 1, which included the effects of energy loss from the system, composed of the electron gas and the electronically excited states of potassium due to radiation from the excited potassium atoms. In addition, the pulsed technique used to measure the conductivity in response to a step function application of the electric field made possible the determination of the relaxation times for the ionization process.

The purpose of the present note is to present more conductivity measurements for the argon-potassium system, which extend the range of neutral gas temperatures and seed concentrations investigated. In addition, measurements of conductivity and relaxation times for the helium-potassium system have been made. The apparatus and experimental techniques were identical, except for a few refinements, to those described in Ref. 1. The calculation scheme was also the same except for re-estimation of radiation losses for the helium-potassium system and appropriate modification of Eqs. (7-12) to include atomic species of differing masses.

Consider first the conductivity measurements made in the argon-potassium system. Experimental and calculated values of conductivity are presented as a function of the current density in Figs. 1-3. Calculated values of the electron temperature T_e are also indicated on the curves. The conductivity measurements presented here are values obtained with neutral gas temperatures T_a of 1500° ± 30°K and 2000° ± 100°K; in all cases, the pressure is 1 atm. In general, the data shown here agree well with the calculated values. In addition to these data, similar results have been obtained for temperatures of 1750° and 1250°K, and measurements at 2000°K were extended to include a mole fraction of 0.1%. The agreement with calculated values and scatter in the data is similar to that for the data shown here.

Note that radiation is the dominant loss mechanism in the low current range, and that measured and calculated values are still in good agreement down to the lowest current densities shown in the figures, about 0.2 and 0.4 amp/cm² for the 1500° and 2000°K gas temperatures, respectively. Hence, there is no reason to suspect that the two-temperature model used in

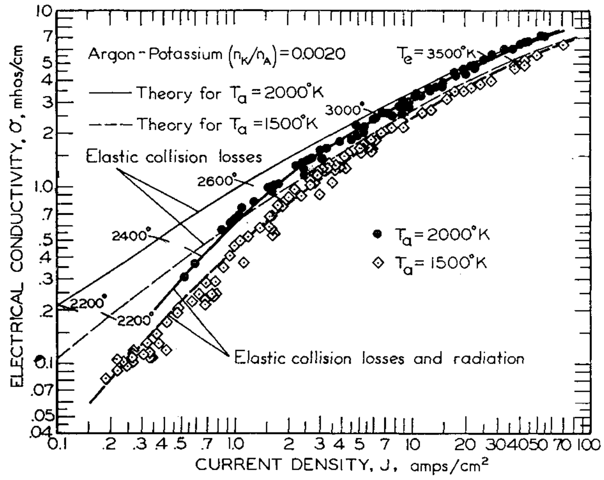

Fig. 1 Dependence of steady-state conductivity on current density.

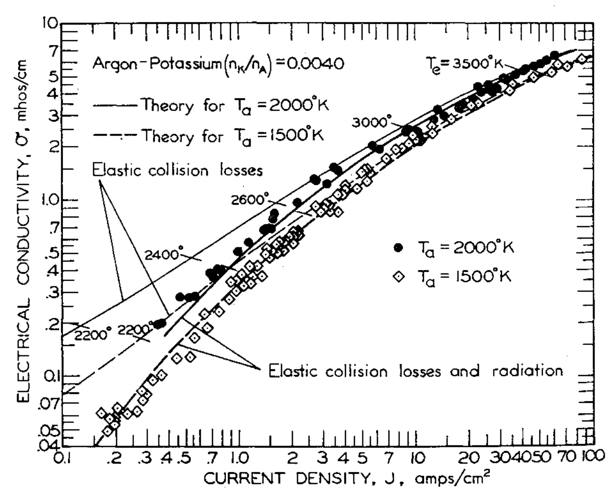

Fig. 2 Dependence of steady-state conductivity on current density.

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